# GENERAL FULL-WAVE GREEN'S FUNCTIONS IN SPECTRAL DOMAIN FOR ARBITRARILY MULTILAYERED DIELECTRIC MEDIA

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## Abstract

**In this paper, we present general analytical formulas of the full-wave spectral domain Green's functions for arbitrarily multilayered dielectric media. The Green's functions are two scalar potentials created by a horizontal dipole, corresponding two electromagnetic modes, transverse magnetic and transverse electric modes to the normal direction of the layer plane. The analytical formulas in spectral domain have simple form and are applicable to arbitrary number of the dielectric layers. The derivation of the formulas employs a technique developed for the derivation of the electrostatic Green's function in multilayered media. The Green's functions used in the spectral domain approach are also presented.**

## I. Introduction

How to determine a Green's function is one of the most important and indispensable step in solving the integral equations formulated by the boundary integral equation techniques such as the boundary element method (BEM) [1]-[4] or the partial-boundary element method (*p*-BEM) [5] and the spectral-domain approach (SDA) [6]-[10]. For free space or for an unbounded homogenous region, the Green's function can be easily obtained and is of simple closed form for both electrostatic and time-harmonic fields. For a one-layered or two-layered structure, the method of images can be employed to obtain the Green's function, with which the Green's function is expressed in a single or double infinite summation of all distributions of the image sources [3]. On the other hand, recent development of the microwave integrated circuits (MICs) and monolithic microwave integrated circuits (MMICs) brings a new trend in the design of the microwave circuits that use a multilayered medium structure in order to provide high cost performance as commercial products [11]-[12]. The efficient analysis of such multilayered medium structure usually requires a Green's function that incorporates partially or all the boundary conditions in the multilayered structures [1], [5]-[10].

It is always to be expected to have an analytical expression for the Green's function, if the expression exists and is possible to derive. The analytical expression can help one to make high computation efficiency or to introduce some techniques to improve the computation efficiency. Moreover, the analytical expression is necessary and indispensable in treating the singularity of the Green's function and its boundary integrals [1], [3].

In spite of the need of the analytical Green's function, for a multilayered media with an arbitrary number of layers, it has not yet been developed, to author's best knowledge. Fortunately, we have developed a general analytical solution of the electrostatic Green's function for multilayered media [13], [14]. This success indicates the possibility of deriving an analytical expression for the full-wave Green's function in the multi-

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layered media. In this paper, using the similar derivation technique, we discovered such full-wave Green's function in the spectral domain. In the derivation, we use two potentials, which were introduced by Harrington [2] and widely used in the spectral domain approach [6]. The potentials are corresponding to two electromagnetic modes, transverse magnetic and transverse electric modes to the normal direction of layer plane, not the modes that are transverse to the direction of the propagation as usually used in the analysis of waveguides and transmission lines. After a brief description of the potentials and the field expressions, we present two general analytical expressions of the Green's function. Since the derivation process is fairly complicated, the detail is omitted here. The Green's functions employed in the spectral domain approach (SDA) are also presented as an application of the derived general formulas.

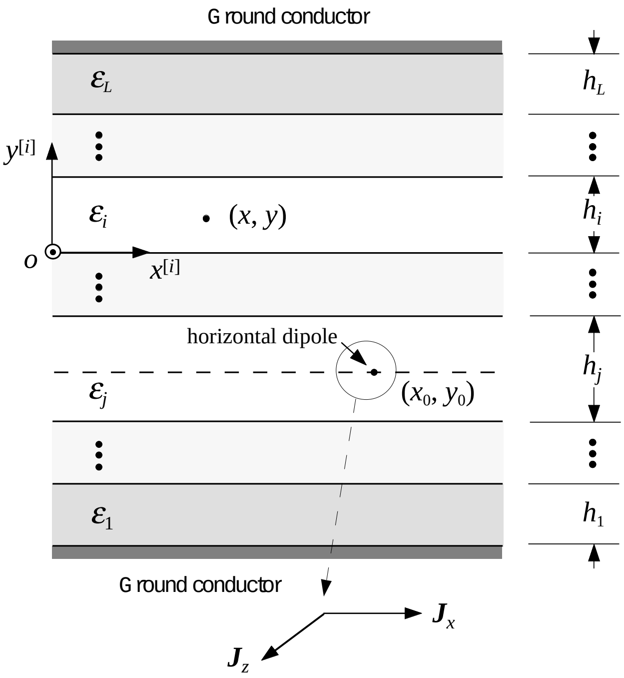


Fig. 1 A two-dimensional multilayered medium structure with a horizontal dipole

## II. Green's Functions for Arbitrarily Multilayered Media

As illustrated in Fig. 1, the arbitrarily multilayered medium structure consists of $L$ isotropic dielectric layers with electric parameters $\varepsilon_i$ ($i$ = 1, 2, ..., $L$) and perfectly conducting ground conductors. The source point and the observation point are placed in $j$th and $i$th layer and denoted by $(x_0, y_0)$ and $(x, y)$, respectively. The source is a line horizontal dipole for a two-dimensional problem. Coordinate systems for analysis are built in each local layer for convenience of analysis. Hence, the $y_0$ and $y$ in such local coordinate systems should have the values in the region $0 \le y_0 \le h_j$ and $0 \le y \le h_i$, respectively. In this paper, we define a layer as "the source layer" when the linesource exists in that layer, or as "the non-source layer" otherwise.

Here, we introduce two scalar potentials $\phi^e$ and $\phi^h$, which satisfy wave equations

$$\nabla^2 \begin{Bmatrix} \phi^e \\ \phi^h \end{Bmatrix} + \left(k^2 - \beta^2\right) \begin{Bmatrix} \phi^e \\ \phi^h \end{Bmatrix} = 0 \qquad (1)$$

in a homogeneous source-free region, where $k^2 = \omega^2 \mu\varepsilon$, and the time convention $\exp(-j\omega t)$ and the $z$ dependence $\exp(-j\beta z)$ are implied. These potentials correspond to two electromagnetic field modes, transverse magnetic to $y$ direction (TM-to-y) and transverse electric to $y$ direction (TE-to-y) [2]. To solve the potentials in the spectral domain, we take the Fourier transform of them as

$$\tilde{\phi}(\alpha, y) = \int_{-\infty}^{\infty} \phi(x, y)\, e^{j\alpha x}\, dx \qquad (2)$$

The Fourier transformed wave equation is then expressed as

$$\left(-\alpha^2 + \frac{\partial^2}{\partial y^2} - \beta^2\right)\tilde{\phi} + k^2\tilde{\phi} = 0 \qquad (3)$$

and its solution is given by

$$\tilde{\phi} = A \cosh \gamma y + B \sinh \gamma y \qquad (4)$$

where $-\gamma^2 = k^2 - \alpha^2 - \beta^2$.

Defining following parameters for convenience of derivation

$$Z^e = \frac{j\gamma}{\omega\varepsilon}, \quad Y^h = \frac{j\gamma}{\omega\mu}, \text{ and}$$

$$Y^e = \frac{1}{Z^e}, \quad Z^h = \frac{1}{Y^e} \qquad (5)$$

we can express all field components as follows:

$$\tilde{E}_y = j\frac{\alpha^2+\beta^2}{\gamma} Z^e \tilde{\phi}^e$$
$$\tilde{H}_y = j\frac{\alpha^2+\beta^2}{\gamma} Y^h \tilde{\phi}^h \tag{6}$$

$$\tilde{E}_x = j\alpha Z^e \frac{\partial\tilde{\phi}^e}{\partial\bar{y}} - j\beta\tilde{\phi}^h$$
$$\tilde{E}_z = j\beta Z^e \frac{\partial\tilde{\phi}^e}{\partial\bar{y}} + j\alpha\tilde{\phi}^h \tag{7}$$

$$\tilde{H}_x = j\beta\tilde{\phi}^e + j\alpha Y^h \frac{\partial\tilde{\phi}^h}{\partial\bar{y}}$$
$$\tilde{H}_z = -j\alpha\tilde{\phi}^e + j\beta Y^h \frac{\partial\tilde{\phi}^h}{\partial\bar{y}} \tag{8}$$

where $\bar{y} = \gamma y$.

Applying such solutions to multilayered media, we have solutions in each homogeneous layer and after a fairly complicated derivation, we find following general formulas for these two potentials.

For TM-to-y mode

$$Z_i^e \frac{\partial\tilde{\phi}_i^e(\bar{y},\bar{y}_0)}{\partial\,\bar{y}} = -\frac{j\tilde{s}_e}{\Delta_e}\begin{cases}\Delta_e^{i+}\left(\bar{h}_i-\bar{y}\right)\Delta_e^{j-}\left(\bar{y}_0\right) & i>j \text{ or } i=j,\, y\ge y_0\\ \Delta_e^{i-}\left(\bar{y}\right)\Delta_e^{j+}\left(\bar{h}_j-\bar{y}_0\right) & i<j \text{ or } i=j,\, y\le y_0\end{cases} \tag{9}$$

For TE-to-y mode

$$\tilde{\phi}_i^h(\bar{y},\bar{y}_0) = \frac{j\tilde{s}_h}{\Delta_h}\begin{cases}\Delta_h^{i+}\left(\bar{h}_i-\bar{y}\right)\Delta_h^{j-}\left(\bar{y}_0\right) & i>j \text{ or } i=j,\, y\ge y_0\\ \Delta_h^{i-}\left(\bar{y}\right)\Delta_h^{j+}\left(\bar{h}_j-\bar{y}_0\right) & i<j \text{ or } i=j,\, y\le y_0\end{cases} \tag{10}$$

where,

$$\tilde{s}_e = \frac{1}{\alpha^2+\beta^2}\left(\alpha\tilde{J}_x+\beta\tilde{J}_z\right) \tag{11.1}$$

$$\tilde{s}_h = \frac{1}{\alpha^2+\beta^2}\left(-\beta\tilde{J}_x+\alpha\tilde{J}_z\right) \tag{11.2}$$

$$\bar{h}_i = \gamma_i h_i\,,\quad i = 1, 2, \cdots, L$$
$$\bar{y} = \gamma_i y\,,\quad \bar{y}_0 = \gamma_i y_0 \tag{12}$$

$$\Delta_m = \Delta_m\left(\bar{h}_1, \bar{h}_2, \cdots, \bar{h}_L\right) \tag{13}$$

$$\Delta_m^{i+}(\bar{y}) = \Delta_m\left(0, 0, \cdots, \bar{y}, \overline{h_{i+1}}, \cdots, \bar{h}_L\right)$$
$$\Delta_m^{i-}(\bar{y}) = \Delta_m\left(\bar{h}_1, \bar{h}_2, \cdots, \overline{h_{i-1}}, \bar{y}, 0, \cdots, 0\right) \tag{14}$$

$$\Delta_m\left(z_1, z_2, \cdots, z_L\right) = \prod_{k=1}^{L}\frac{1}{Y_k^m}\cdot\prod_{k=1}^{L-1}\left(Y_k^m\frac{\partial}{\partial z_k}+Y_{k+1}^m\frac{\partial}{\partial z_{k+1}}\right)\cdot\prod_{k=1}^{L}\sinh z_k \tag{15}$$

$$m = e, h$$

## III. Green's Functions in Spectral Domain Approach (SDA)

In the spectral domain approach, Green's function or a matrix to link the electric field and the source is necessary. From above general formulas given in (9), (10), we can easily derive such matrix.

For $i > j$ or $i < j$, we have corresponding terms in the Green's function given as follows:

$$\tilde{G}_e^{ij+} = \frac{\Delta_e^{i+}\left(\bar{h}_i-\bar{y}\right)\Delta_e^{j-}\left(\bar{y}_0\right)}{\Delta_e} \tag{16.1}$$

$$\tilde{G}_e^{ij-} = \frac{\Delta_e^{i-}\left(\bar{y}\right)\Delta_e^{j+}\left(\bar{h}_j-\bar{y}_0\right)}{\Delta_e} \tag{16.2}$$

$$\tilde{G}_h^{ij+} = \frac{\Delta_h^{i+}\left(\bar{h}_i-\bar{y}\right)\Delta_h^{j-}\left(\bar{y}_0\right)}{\Delta_h} \tag{16.3}$$

$$\tilde{G}_h^{ij-} = \frac{\Delta_h^{i-}\left(\bar{y}\right)\Delta_h^{j+}\left(\bar{h}_j-\bar{y}_0\right)}{\Delta_h} \tag{16.4}$$

Therefore, substituting these terms into Eq. (6) gives

$$\begin{bmatrix} \tilde{E}_x^{ij} \\ \tilde{E}_z^{ij} \end{bmatrix} = \begin{bmatrix} \tilde{G}_{xx}^{ij} & \tilde{G}_{xz}^{ij} \\ \tilde{G}_{zx}^{ij} & \tilde{G}_{zz}^{ij} \end{bmatrix} \cdot \begin{bmatrix} \tilde{J}_x^{j} \\ \tilde{J}_z^{j} \end{bmatrix} \quad (17)$$

where the elements of the matrix are generally given by

$$\tilde{G}_{xx}^{ij} = \frac{1}{\alpha^2 + \beta^2}\left(\alpha^2 \tilde{G}_e^{ij} - \beta^2 \tilde{G}_h^{ij}\right) \quad (18.1)$$

$$\tilde{G}_{xz}^{ij} = \tilde{G}_{zx}^{ij} = \frac{\alpha\beta}{\alpha^2 + \beta^2}\left(\tilde{G}_e^{i} + \tilde{G}_h^{ij}\right) \quad (18.2)$$

$$\tilde{G}_{zz}^{ij} = \frac{1}{\alpha^2 + \beta^2}\left(\beta^2 \tilde{G}_e^{ij} - \alpha^2 \tilde{G}_h^{ij}\right) \quad (18.3)$$

These formulas are more general expressions of the spectral domain Green's function functions than those given in [1], [6].

## IV. Conclusion

We have presented general analytical formulas of the full-wave Green's functions in spectra domain for arbitrarily multilayered dielectric media. The Green's functions are two scalar potentials created by a horizontal dipole, corresponding two electromagnetic modes, transverse magnetic and transverse electric modes to the normal direction of layer plane. These formulas have simple form and are applicable to arbitrary number of the dielectric layers. The derivation of the formulas uses a technique developed for the derivation of the electrostatic Green's function in multilayered media. The Green's functions used in the spectral domain approach are also presented. One obvious merit of these formulas is that they are given in final analytical expressions and the expressions are true for arbitrary number of layers. Being analytical and applicable to arbitrary number of layers will be helpful to develop efficient calculation techniques of the Green's function based on those expressions, for example, to extract the singular part from the Green's function, to introduce an approximate formula that may be more suitable to the numerical computation, and to develop a general computation program for an arbitrarily multilayered medium structure. These analytical formulas are able to be extended to the solutions for three-dimensional Green's functions, since the derivation procedure after expanding the Green's function in a two-dimensional plane is completely the same as that of two-dimensional problem. We believe that these general analytical formulas for full-wave Green's function presented in this paper will provide a key to open the door of the analysis of the arbitrarily multilayered medium structures.

## References


[1] T. Itoh, Ed., "Numerical techniques for microwave and millimeter-wave passive structures," John Wiley & Sons, Inc., New York, 1989. (Chapter 3: Integral equation technique, Chapter 5: Spectral domain approach)

[2] R. F. Harrington, "Time-harmonic electromagnetic fields," McGraw-Hill, New York, 1961. (Chapter 3: Some theorems and concepts)

[3] R. E. Collin, "Field theory of guided waves," 2nd ed., IEEE Press, New York, 1991. (Chapter 2: Green's functions)

[4] W. Schroeder, et al, IEEE MTT-S Int. Microwave Symp. Dig., June, 1989, pp. 711-714.

[5] K. Atsuki, et al, IEEE MTT-43, No. 5, pp. 1153-1161, May 1995.

[6] T. Itoh, IEEE MTT-26, pp. 983-987, No. 12, Dec. 1978.

[7] T. Itoh, IEEE MTT- 28, pp. 733-736, No. 7, July 1980.

[8] N. K. Das, et al, IEEE MTT-35, No. 3, pp. 326-335, March. 1987.

[9] R. Crampagne, et al, IEEE MTT-26, No. 2, pp. 82-87, Feb. 1978.

[10] W. Delbare, et al, IEEE MTT-37, No. 10, pp. 1562-1568, Oct. 1989.

[11] H. Ogawa, et al, IEEE MTT-S Int. Microwave Symp. Dig., Boston, MA, June 10-14,1990.

[12] T. Tokumitsu, et al, IEICE Trans. Electron., E75-C, No.6, pp. 698-706, June 1992.

[13] K. Li, et al, IEEE MTT-45, No.1 Jan. 1997.

[14] K. Li, et al, IEICE Trans. Electronics, vol. J78-C-I, No.10, pp. 1366-1372, Oct. 1995.